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A perturbation method for the accurate estimation of the vibration spectrum for the Timoshenko beam

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Abstract

In the stabilization and control of the vibration of flexible structures, a knowledge of the system's vibration spectrum is crucial. In the past, we developed a perturbation approach that, when coupled with the asymptotic *wave propagation method* of Keller and Rubinow, yields highly accurate estimates for the vibration spectra of certain beam problems. Here, we extend this method to a stand-alone method, applied to the Timoshenko beam problem; it is seen that the wave propagation method is a special case of this method, as are all other asymptotic results that we have found in the literature.

1. Introduction

The Timoshenko beam equations constitute a model for a thick beam, as they incorporate the effects of rotary inertia and deformation due to shear. An asymptotic estimation of the vibration spectrum of the Timoshenko beam was performed in Ref. [1], and asymptotic expressions for the spectrum also have been derived in Refs. [2–7], with Refs. [5–7] treating the case with coefficients which are non-constant, spatially. However, as these estimates are asymptotic, they are least accurate for the lowest frequencies and, in particular, for the fundamental frequency—the frequencies associated with the highest vibration energies. Here, we extend a perturbation approach, developed for the Euler–Bernoulli [8] and slewing [9] beams, to the case of a clamped–clamped Timoshenko beam. In the process, we see that this method actually is a stand-alone method, for which the zeroth approximation reduces to the asymptotic *wave propagation method* of Keller and Rubinow [10,11], as well as to other results in the literature [2–4]. Of course, these and similar problems are solvable using commercially available software packages. The disadvantage is that these packages generally employ FEM or similar numerical methods, and thus do not offer the analytical and physical insights provided by asymptotic methods. Further, if this method can be made rigorous, it may become part of the arsenal of routines used by commercial packages. Here, although we consider only clamped–clamped geometry, the method applies similarly to any combination of energy-conserving boundary

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conditions. We begin by introducing the Timoshenko beam eigenvalue problem; we then apply the perturbation method to the problem, and we end with examples.

2. The problem

We consider a Timoshenko beam of length L, with both ends strongly clamped. We let $W(x,t)$ be the lateral displacement at point x at time t, and $\Phi(x,t)$ the bending angle at point x at time t. The Timoshenko beam equations, then, are [1–7,12]

$$\rho W_{tt} - KW_{xx} + K\Phi_x = 0, \tag{1}$$

$$I_\rho \Phi_{tt} - EI\Phi_{xx} + K[\Phi - W_x] = 0, \quad 0<x<L. \tag{2}$$

Here, EI is the flexural rigidity, ρ the mass density, K the shear stiffness, I_ρ the rotary inertia, and $W_x = \partial W/\partial x$, etc. We assume that these parameters are constant, and that the beam's cross section is uniform throughout.

The boundary conditions are

$$W(0,t) = \Phi(0,t) = W(L,t) = \Phi(L,t) = 0, \quad t>0. \tag{3}$$

In order to find the eigenfrequencies $-\mathrm{i}\xi^2$, we let

$$W(x,t) = w(x)\mathrm{e}^{-\mathrm{i}\xi^2 t}, \quad \Phi(x,t) = \phi(x)\mathrm{e}^{-\mathrm{i}\xi^2 t}. \tag{4}$$

Then, eliminating ϕ from Eqs. (1) and (2), we have the ODE

$$w^{(4)}(x) + \frac{\rho}{EI}\left(\frac{EI}{K} + \frac{I_\rho}{\rho}\right)\xi^4 w''(x) - \frac{\rho}{EI}\xi^4 w(x) + \frac{\rho}{EI}\frac{I_\rho}{K}\xi^8 w(x) = 0, \quad 0<x<L. \tag{5}$$

Now, following Ref. [12], we introduce dimensionless variables and parameters:

$$\begin{aligned} x_1 &= \frac{x}{L}, \\ w_1 &= \frac{w}{L}, \\ \lambda^2 &= \sqrt{\frac{\rho}{EI}}\, L^2\xi^2, \\ \beta_1 &= \frac{I_\rho}{\rho L^2}, \\ \beta_2 &= \frac{EI}{KL^2}. \end{aligned} \tag{6}$$

In this case, Eq. (5) becomes

$$w_1^{(4)}(x_1) + 2r^2\lambda^4 w_1(x_1) + (s^4\lambda^8 - \lambda^4)w_1(x_1) = 0, \quad 0<x_1<1, \tag{7}$$

where

$$2r^2 = \beta_1 + \beta_2, \quad s^4 = \beta_1\beta_2. \tag{8}$$

Further, the w boundary conditions become, as in Ref. [12],

$$\begin{aligned} w_1(0) &= \beta_2 w_1'''(0) + (1+\beta_2^2\lambda^4)w_1'(0) = 0, \\ w_1(1) &= \beta_2 w_1'''(1) + (1+\beta_2^2\lambda^4)w_1'(1) = 0. \end{aligned} \tag{9}$$

Now we let $w_1 = \mathrm{e}^{\mu x_1}$ in Eq. (7) and find that

$$\mu = \pm\mathrm{i}\lambda^2\gamma_1, \pm\mathrm{i}\lambda^2\gamma_2,$$

where

$$\gamma_1 = \sqrt{r^2 - \sqrt{r^4 - s^4 + 1/\lambda^4}}, \quad \gamma_2 = \sqrt{r^2 + \sqrt{r^4 - s^4 + 1/\lambda^4}}. \tag{10}$$

Applying the boundary conditions equation (9) to the general solution

$$w_1(x) = A\mathrm{e}^{\mathrm{i}\lambda^2\gamma_1 x_1} + B\mathrm{e}^{\mathrm{i}\lambda^2\gamma_2 x_1} + C\mathrm{e}^{-\mathrm{i}\lambda^2\gamma_1 x_1} + D\mathrm{e}^{-\mathrm{i}\lambda^2\gamma_2 x_1} \tag{11}$$

leads to the system

$$\begin{aligned}
&A + B + C + D = 0,\\
&f_1 A + f_2 B - f_1 C - f_2 D = 0,\\
&A\mathrm{e}^{\mathrm{i}\lambda^2\gamma_1} + B\mathrm{e}^{\mathrm{i}\lambda^2\gamma_2} + C\mathrm{e}^{-\mathrm{i}\lambda^2\gamma_1} + D\mathrm{e}^{-\mathrm{i}\lambda^2\gamma_2} = 0,\\
&Af_1\mathrm{e}^{\mathrm{i}\lambda^2\gamma_1} + Bf_2\mathrm{e}^{\mathrm{i}\lambda^2\gamma_2} - Cf_1\mathrm{e}^{-\mathrm{i}\lambda^2\gamma_1} - Df_2\mathrm{e}^{-\mathrm{i}\lambda^2\gamma_2} = 0.
\end{aligned} \tag{12}$$

Here,

$$f_i = -\beta_1\gamma_i^3 + \beta_1^2\gamma_i + \frac{1}{\lambda^4}\gamma_i, \quad i = 1, 2. \tag{13}$$

The eigenfrequencies, then, are determined by the condition that

$$\begin{aligned}
0 = \det M &= \begin{vmatrix}
1 & 1 & 1 & 1\\
f_1 & f_2 & -f_1 & -f_2\\
\mathrm{e}^{\mathrm{i}\lambda^2\gamma_1} & \mathrm{e}^{\mathrm{i}\lambda^2\gamma_2} & \mathrm{e}^{-\mathrm{i}\lambda^2\gamma_1} & \mathrm{e}^{-\mathrm{i}\lambda^2\gamma_2}\\
f_1\mathrm{e}^{\mathrm{i}\lambda^2\gamma_1} & f_2\mathrm{e}^{\mathrm{i}\lambda^2\gamma_2} & -f_1\mathrm{e}^{-\mathrm{i}\lambda^2\gamma_1} & -f_2\mathrm{e}^{-\mathrm{i}\lambda^2\gamma_2}
\end{vmatrix}\\
&= (f_1 - f_2)^2\mathrm{e}^{\mathrm{i}\lambda^2(\gamma_1+\gamma_2)} - (f_1 + f_2)^2\mathrm{e}^{\mathrm{i}\lambda^2(\gamma_1-\gamma_2)} - (f_1 + f_2)^2\mathrm{e}^{\mathrm{i}\lambda^2(\gamma_2-\gamma_1)} + (f_1 - f_2)^2\mathrm{e}^{-\mathrm{i}\lambda^2(\gamma_1+\gamma_2)} + 8f_1f_2.
\end{aligned} \tag{14}$$

3. Perturbation method

Standard asymptotic methods now would neglect all terms of $O(1/\lambda^4)$, as in Ref. [1]. Here, instead, we choose as our perturbation parameter

$$\varepsilon = \frac{1}{\lambda^4}. \tag{15}$$

Then,

$$\begin{aligned}
\gamma_1 = \gamma_1(\varepsilon) &= \sqrt{r^2 - \sqrt{r^4 - s^4 + \varepsilon}}\\
&= \sqrt{r^2 - \sqrt{r^4 - s^4}} - \left[4\sqrt{r^2 - \sqrt{r^4 - s^4}}\sqrt{r^4 - s^4}\right]^{-1}\varepsilon + O(\varepsilon^2)
\end{aligned} \tag{16}$$

and

$$\gamma_2 = \gamma_2(\varepsilon) = \sqrt{r^2 + \sqrt{r^4 - s^4}} + \left[4\sqrt{r^2 + \sqrt{r^4 - s^4}}\sqrt{r^4 - s^4}\right]^{-1}\varepsilon + O(\varepsilon^2). \tag{17}$$

Using Eq. (8), we have

$$\sqrt{r^4 - s^4} = \tfrac{1}{2}|\beta_1 - \beta_2|. \tag{18}$$

Thus, upon simplification, for $\beta_1 > \beta_2$, Eqs. (13) and (14) become, respectively,

$$\gamma_1(\varepsilon) = \sqrt{\beta_2} + \left[2\sqrt{\beta_2}\,(\beta_2 - \beta_1)\right]^{-1}\varepsilon + O(\varepsilon^2) \tag{19}$$

and

$$\gamma_2(\varepsilon) = \sqrt{\beta_1} + \left[2\sqrt{\beta_1}\,(\beta_1 - \beta_2)\right]^{-1}\varepsilon + O(\varepsilon^2). \tag{20}$$

Similarly, for $\beta_1 < \beta_2$, we just interchange the two expressions. Thus, WLOG, we consider the case $\beta_1 > \beta_2$. We use Eqs. (19) and (20) to rewrite Eq. (13) as

$$f_1(\varepsilon) = \alpha_1\alpha_2^2(\alpha_2^2 - \alpha_1^2) + \frac{2\alpha_1^4 - 5\alpha_1^2\alpha_2^2 + \alpha_2^4}{2\alpha_1(\alpha_1^2 - \alpha_2^2)}\varepsilon + O(\varepsilon^2)$$

and

$$f_2(\varepsilon) = \frac{\alpha_1^2\alpha_2}{\alpha_1^2 - \alpha_2^2}\varepsilon + O(\varepsilon^2), \tag{21}$$

where $\alpha_1 = \sqrt{\beta_1}$ and $\alpha_2 = \sqrt{\beta_2}$. Further, we expand the eigenvalues as

$$\lambda = \lambda_0 + \lambda_1\varepsilon + O(\varepsilon^2) \tag{22}$$

and, upon simplification, we have

$$e^{\pm i\lambda^2\gamma_j} = e^{\pm i\lambda_0^2\alpha_j}\left\{1 \pm \varepsilon i\left[2\lambda_0\lambda_1\alpha_i + \frac{\lambda_0^2}{2\alpha_1(\alpha_1^2 - \alpha_2^2)}\right]\right\} + O(\varepsilon^2), \quad j = 1, 2. \tag{23}$$

Finally, after very much, but straightforward, simplification, we may write the determinant equation (14) as

$$\begin{aligned}
D(\varepsilon) = {} & \det M(\varepsilon) \\
= {} & C_1[e^{i\lambda_0^2(\alpha_1+\alpha_2)} - e^{i\lambda_0^2(\alpha_1-\alpha_2)} - e^{i\lambda_0^2(\alpha_2-\alpha_1)} + e^{-i\lambda_0^2(\alpha_1+\alpha_2)}] \\
& + \varepsilon\left\{\left[2(C_2 - C_3) + 2iC_1\lambda_0\lambda_1(\alpha_1 + \alpha_2) - iC_1\frac{\lambda_0^2}{2\alpha_1\alpha_2(\alpha_1 + \alpha_2)}\right]e^{i\lambda_0^2(\alpha_1+\alpha_2)}\right. \\
& + \left[-2(C_2 + C_3) - 2iC_1\lambda_0\lambda_1(\alpha_1 - \alpha_2) - iC_1\frac{\lambda_0^2}{2\alpha_1\alpha_2(\alpha_1 - \alpha_2)}\right]e^{i\lambda_0^2(\alpha_1-\alpha_2)} \\
& + \left[-2(C_2 + C_3) + 2iC_1\lambda_0\lambda_1(\alpha_1 - \alpha_2) + iC_1\frac{\lambda_0^2}{2\alpha_1\alpha_2(\alpha_1 - \alpha_2)}\right]e^{i\lambda_0^2(\alpha_2-\alpha_1)} \\
& \left. + \left[2(C_2 - C_3) - 2iC_1\lambda_0\lambda_1(\alpha_1 + \alpha_2) + iC_1\frac{\lambda_0^2}{2\alpha_1\alpha_2(\alpha_1 + \alpha_2)}\right]e^{-i\lambda_0^2(\alpha_1+\alpha_2)} - 8\alpha_1^3\alpha_2^3\right\} + O(\varepsilon^2).
\end{aligned} \tag{24}$$

Here,

$$C_1 = \alpha_1\alpha_2^2(\alpha_2^2 - \alpha_1^2), \quad C_2 = \frac{2\alpha_1^4 - 5\alpha_1^2\alpha_2^2 + \alpha_2^4}{2\alpha_1(\alpha_1^2 - \alpha_2^2)}, \quad C_3 = \frac{\alpha_1^2\alpha_2}{\alpha_1^2 - \alpha_2^2}.$$

We now require that the coefficients of ε^0 and ε^1 in Eq. (24) be zero; thus λ_0 must satisfy

$$e^{i\lambda_0^2(\alpha_1+\alpha_2)} - e^{-i\lambda_0^2(\alpha_1-\alpha_2)} - e^{i\lambda_0^2(\alpha_2-\alpha_1)} + e^{-i\lambda_0^2(\alpha_1+\alpha_2)} = 0, \tag{25}$$

then λ_1 is determined from Eq. (24) to be

$$\lambda_1 = \frac{f(\lambda_0)}{g(\lambda_0)},$$

where

$$f(\lambda_0) = -2C_1[(C_2-C_3)e^{i\lambda_0^2(\alpha_1+\alpha_2)} - (C_2+C_3)e^{i\lambda_0^2(\alpha_2-\alpha_1)} - (C_2+C_3)e^{i\lambda_0^2(\alpha_1-\alpha_2)} + (C_2-C_3)e^{-i\lambda_0^2(\alpha_1+\alpha_2)}]$$
$$+\frac{iC_1^2\lambda_0^2}{2\alpha_1\alpha_2}\left[\frac{1}{\alpha_1+\alpha_2}e^{i\lambda_0^2(\alpha_1+\alpha_2)} + \frac{1}{\alpha_2-\alpha_1}e^{i\lambda_0^2(\alpha_2-\alpha_1)} + \frac{1}{\alpha_1-\alpha_2}e^{i\lambda_0^2(\alpha_1-\alpha_2)} - \frac{1}{\alpha_1+\alpha_2}e^{-i\lambda_0^2(\alpha_1+\alpha_2)}\right]$$

and

$$g(\lambda_0) = 2iC_1^2\lambda_0[(\alpha_1+\alpha_2)e^{i\lambda_0^2(\alpha_1+\alpha_2)} + (\alpha_1-\alpha_2)e^{i\lambda_0^2(\alpha_2-\alpha_1)} + (\alpha_2-\alpha_1)e^{i\lambda_0^2(\alpha_1-\alpha_2)} - (\alpha_1+\alpha_2)e^{-i\lambda_0^2(\alpha_1+\alpha_2)}].$$

Finally, we set

$$\lambda^{(0)} = \lambda_0, \quad \varepsilon = \varepsilon_0 = \frac{1}{(\lambda^{(0)})^4},$$

as the zeroth approximation. The first approximation, then, is

$$\lambda^{(1)} = \lambda_0 + \varepsilon_0\lambda_1.$$

We continue inductively, setting

$$\varepsilon_n = \frac{1}{(\lambda^{(n)})^4}$$

and updating the approximation via

$$\lambda^{(n+1)} = \lambda_0 + \varepsilon_n\lambda_1. \tag{26}$$

It is easily seen that Eq. (25) is equivalent to

$$\sin\lambda_0^2\alpha_1 \sin\lambda_0^2\alpha_2 = 0. \tag{27}$$

This expression, in turn, essentially is the expression derived in Ref. [4] and, if modified to the free–free problem, will give the results in Refs. [2,3]. Expression (26) leads to the two branches

$$\lambda_0^2 = \frac{n\pi}{\alpha_1} = n\pi L\sqrt{\frac{\rho}{I_\rho}}, \quad n = 1,2,\ldots, \tag{28}$$

$$\lambda_0^2 = \frac{n\pi}{\alpha_2} = n\pi L\sqrt{\frac{K}{EI}}, \quad n = 1,2,\ldots\ . \tag{29}$$

The corresponding dimensional values, from the fact that $\xi^2 = (1/L^2)\sqrt{(EI/\rho)}\lambda^2$ (Eq. (6)), are

$$\xi_0^2 = \frac{n\pi}{L}\sqrt{\frac{EI}{I_\rho}}, \quad n = 1,2,\ldots, \tag{30}$$

$$\xi_0^2 = \frac{n\pi}{L}\sqrt{\frac{K}{\rho}}, \quad n = 1,2,\ldots, \tag{31}$$

which are seen to be equivalent to the branches Λ_2 and Λ_1, respectively, in Ref. [1] (where, as the right end is clamped, we let $\alpha = \beta = \infty$). Further, they are equivalent to the expressions given in Refs. [5–7], when the coefficients therein are constant.

The choice of parameters for our examples is restricted by the fact that, from Eq. (9) we need to have

$$r^4 - s^4 = \frac{1}{4}(\beta_1-\beta_2)^2 \gg \frac{1}{\lambda_0^4}$$

(correcting a minor error in Ref. [1]), i.e., that

$$\frac{1}{4}\left(\frac{EI}{K} - \frac{I_\rho}{\rho}\right)^2 \frac{\rho}{I_\rho}\frac{n^2\pi^2}{L^2} \gg 1 \tag{32}$$

and

$$\frac{1}{4}\left(\frac{EI}{K}-\frac{I_\rho}{\rho}\right)^2\frac{K}{EI}\frac{n^2\pi^2}{L^2}\gg 1 \tag{33}$$

for the first and second branches, respectively.

4. Examples

We present two sets of examples. The first set uses "simple" values for the parameters in order to illustrate the method, and to allow both branches to be exhibited clearly, while the second set uses the parameter values for a typical steel beam, including realistic values for the beam's length.

To begin, then, we illustrate the perturbation method using the same parameters that were used in the first example in Ref. [1]. However, we allow the length L to vary, in order to investigate what happens vis-a-vis the restrictions in Eqs. (31) and (32). Thus we choose $\rho=1$, $I=3$, $E=2.5$ and $K=1.5$, and consider the three cases $L=.01$, .1 and .5. We compare the perturbation results with those resulting from applying the Legendre-tau spectral method to the results. This method entails transforming the problem equations (1)–(3) to one on the interval $-1\leqslant x\leqslant 1$, letting $W(x,t)=e^{\Lambda t}w(x)$ and $\Phi(x,t)=e^{\Lambda t}\phi(x)$, and setting

$$w(x)=\sum_{n=0}^{N}a_nP_n(x),$$

$$\phi(x)=\sum_{n=0}^{N}b_nP_n(x).$$

Here, P_n is the Legendre polynomial of degree n. Computations at $n=100$ and 110 show that the first 50 eigenfrequencies converge to at least nine decimal places at $n=100$.

In Table 1, we give comparisons between the numerical (N) and 0th-order perturbation (P) results, for $L=.01$, .1 and .5. For $L=.01$, we see excellent agreement, to at least six decimal places, even at the low end of the spectrum. As expected from Eqs. (31) and (32), the 0th-order perturbation results become less accurate as L increases; so, for $L=.5$, we have only three-decimal-place agreement for the lowest frequencies. Table 2, then, gives a comparison between the numerical results and the perturbation results for the first seven frequencies for the case $L=.5$, that is, for those frequencies with agreement to less than five decimal places in Table 1. Here, we see that the updates $(\xi^{(n)})^2=(1/L^2)\sqrt{(EI/\rho)}(\lambda^{(n)})^2$ give exact agreement for the last four results and near agreement for the second and third frequencies. We also see that, for the lowest frequency, the

Table 1
The first 10 frequencies for $L=.01$, .1 and .5

$L=.01$		$L=.1$		$L=.5$		B
N	P	N	P	N	P	
384.7649	384.7649	38.4759	38.4765	7.6912	7.6953	II
608.3671	608.3670	60.8398	60.8367	12.183	12.167	I
769.5301	769.5300	76.9547	76.9530	15.399	15.391	II
1154.294	1154.295	115.421	115.430	23.085	23.086	II
1216.735	1216.734	121.684	121.673	24.341	24.335	I
1539.060	1539.060	153.904	153.906	30.783	30.781	II
1825.100	1825.100	182.514	182.510	36.504	36.502	I
1923.825	1923.825	192.381	192.383	38.476	38.476	II
2308.5897	2308.5897	230.858	230.858	46.172	46.172	II
2433.4674	2433.4672	243.347	243.347	48.669	48.669	I

In each case, the numerical (N) frequency appears first, followed by the 0th-order perturbation (P) frequency $\xi_0^2=(1/L^2)\sqrt{(EI/\rho)}\lambda_0^2$. I signifies that the frequency is from the branch given by Eq. (29); II, that it is from Eq. (30).

Table 2
The first seven frequencies for $L = .5$

N	P_0	P_1	P_2	P_3	P_4	P_5
7.6912	7.6953	7.6934	7.6925	7.6921	7.6920	7.6920
12.183	12.167	12.177	12.181	12.181		
15.399	15.391	15.396	15.398	15.398		
23.085	23.086	23.085				
24.341	24.335	24.339	24.340	24.341		
30.783	30.781	30.783				
36.504	36.502	36.504				

The first column gives the numerical (N) results, while subsequent columns give the perturbation approximations $P_n = (\xi^{(n)})^2$.

Table 3
The frequencies from each branch Eqs. (29) and (30), for the values $n = 1, \ldots, 10, 20, 30, 40, 50$ therein, for the cases $L = 1$ and 3

n	$L = 1$				$L = 3$			
	Branch 1		Branch 2		Branch 1		Branch 2	
	N	P	N	P	N	P	N	P
1	6.1143	6.0837	3.8395	3.8477	2.1035	2.0279	1.2588	1.2826
2	12.261	12.167	7.7126	7.6953	4.2512	4.0558	2.6286	2.5651
3	18.288	18.251	11.472	11.543	6.1872	6.0837	3.7203	3.8477
4	24.358	24.335	15.369	15.391	8.1802	8.1116	5.0700	5.1302
5	30.424	30.418	19.224	19.238	10.144	10.139	6.3729	6.4128
6	36.521	36.502	23.076	23.086	12.222	12.167	7.6662	7.6953
7	42.600	42.586	26.927	26.934	14.238	14.195	8.9578	8.9779
8	48.681	48.669	30.788	30.781	16.257	16.223	10.292	10.260
9	54.766	54.753	34.620	34.629	18.290	18.251	11.516	11.543
10	60.847	60.837	38.470	38.477	20.310	20.279	12.805	12.826
20	121.68	121.67	76.949	76.953	40.574	40.558	25.640	25.651
30	182.51	182.51	115.43	115.43	60.847	60.837	38.465	38.477
40	243.35	243.35	153.90	153.91	81.125	81.116	51.298	51.302
50	304.19	304.18	192.38	192.38	101.40	101.39	64.124	64.128

As in Table 1, the numerical (N) results are compared to the 0th-order perturbation (P) results.

Table 4
The first three frequencies from each branch, for $L = 3$

N	Best approximation
2.1035	$2.0820 = P_8$
4.2512	$4.2112 = P_{13}$
6.1872	$6.1414 = P_8$
1.2588	$1.2671 = P_6$
2.6286	$2.6151 = P_8$
3.7203	$3.7523 = P_{11}$

The first column gives the numerical (N) results, while the second column gives the number to which the perturbation approximations converge. Again, $P_n = (\xi^{(n)})^2$, the nth perturbation update.

updates give much more accurate results than the 0th-order approximation. Next, we test the limitations of the method by looking at what happens for $L = 1$ and 3. Table 3 is similar to Table 1, in that it lists the numerical and perturbation results for $L = 1$ and 3. However, here we separate the two branches, and we list the frequencies for $n = 1, \ldots, 10, 20, 30, 40$ and 50 (for n given in Eqs. (30) and (31)). Here we see as little as one

decimal place agreement for a few of the lower entries (for $L=3$, $n=1,2$ on Branch I and $n=3$ on Branch II), although, as expected, there is much better agreement for higher values of n. In Table 4, then, we apply the perturbations to each of the $n=1,2,3$ frequencies for $L=3$. We see that the perturbation method gives much improved approximation for these lowest frequencies. We have performed these same calculations for various larger values of L. For $L=10$, the accuracy of the perturbation approximation is not much different than for the case $L=3$. However, for $L\geqslant 20$, the approximations become progressively worse, as expected, although, asymptotically, there still is good agreement for n large enough vis-a-vis Eqs. (32) and (33).

Next, we consider a steel beam as described in Ref. [13], with modulus of elasticity $E=3.0\times 10^7$ psi (lb/in^2) and shear stiffness $K=9.84\times 10^6$ lb in/s^2. (Note: We use the British system because (a) in c-g-s, the very large values of E and K result in unreliable numerical results, while (b) in m-k-s, the lengths will be of the same order as those in the examples above.)

We treat a solid, circular cylindrical beam with a radius of $r=2$ in, linear mass density $\rho=3.56$ lb/in and resulting moment of inertia of $I=2.01\times 10^2$ in^4. We consider the three cases $L=10$, 20 and 30 in, thus, with length-per-radius ratios $L/r=5,10$ and 15, respectively. Numerical computations at $n=100$ and 110 show that the first 50 eigenfrequencies converge to at least seven decimal places at $n=100$.

Table 5
The first 15 frequencies, along with the 21st, 28th, 35th and 42nd frequencies, for $L=10$, 20 and 30

$L=10$		$L=20$		$L=30$		B
N	P	N	P	N	P	
521.75	522.58	259.66	261.29	171.77	174.19	II
1042.8	1045.2	518.12	522.58	342.07	348.38	II
1566.4	1567.7	781.16	783.86	518.58	522.58	II
2089.1	2090.3	1042.8	1045.2	693.32	696.77	II
2612.0	2612.9	1304.7	1306.4	868.36	870.96	II
3134.3	3135.5	1565.8	1567.7	1042.6	1045.2	II
3262.3	3234.6	1671.6	1617.3	1157.5	1078.2	I
3657.4	3658.0	1827.7	1829.0	1217.4	1219.3	II
4180.1	4180.6	2089.3	2090.3	1392.0	1393.5	II
4702.7	4703.2	2350.6	2351.6	1566.2	1567.7	II
5225.3	5225.8	2612.0	2612.9	1740.6	1741.9	II
5747.9	5748.3	2873.3	2874.2	1914.9	1916.1	II
6270.5	6270.9	3134.7	3135.4	2089.2	2090.3	II
6482.8	6469.2	3261.8	3234.6	2197.0	2156.4	I
6793.2	6793.5	3396.1	3396.7	2263.6	2264.5	II
9712.9	9703.7	4870.1	4851.9	3261.9	3234.6	I
12 945.0	12938.0	6482.8	6469.2	4333.2	4312.8	I
16 178.0	16173.0	8097.4	8086.4	5399.5	5391.0	I
19 412.0	19407.0	9712.9	9703.7	6582.9	6469.2	I

In each case, the numerical (N) frequency appears first, followed by the 0th-order perturbation (P) frequency $\xi_0^2=(1/L^2)\sqrt{(EI/\rho)}\lambda_0^2$. I signifies that the frequency is from the branch given by Eq. (30); II by Eq. (31).

Table 6
The first frequency from each branch for the example with results given in Table 5

$L=10$		$L=20$		$L=30$	
N	B	N	B	N	B
521.75	$521.98=P_3$	259.66	$260.28=P_5$	171.77	$172.57=P_6$
3262.3	$3254.2=P_7$	1671.6	$1646.6=P_{10}$	1157.5	$1125.1=P_{12}$

Again, N represents the numerical result, and B the "best approximation" from the perturbation results $P_n=(\xi^{(n)})^2$.

The format of Table 5 is identical to that of Table 1, except that we include the first 15 frequencies and, in addition, the next four frequencies from branch I (as the majority of these lower frequencies are from branch II).

We see that, for $L = 10$, there is agreement in two decimal places for the very lowest frequencies on each branch, improving to three and four decimal places for the remaining frequencies. This agreement seems to get progressively worse as the length increases, suggested by our earlier set of examples and by the results in Ref. [1]. Of course, the greater the ratio L/r, the more accurate the prediction of the simpler Euler–Bernoulli model become.

Finally, in Table 6, we apply the perturbations to the lowest frequency from each branch, for each of the three cases $L = 10, 20$ and 30. We see here, too, that the perturbations lead to much improved estimations for these lowest frequencies.

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